PHILOSOPHICAL TRANSACTIONS.

I. *On the Laws of the Rise and Fall of the Tide in the River Thames.*

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In the winter of 1840–1841, an extensive series of observations of tides was made, in accordance with my suggestions, at Deptford Royal Victualling Yard. For these observations, as well as for those which follow (and which form more immediately the subject of this paper), I am indebted to Captain Shirreff, R.N., Captain Superintendent of the Royal Victualling Yard and Dock Yard at Deptford. By the kindness of this able officer, I was allowed to give such directions to the police constables on duty in the Victualling Yard as I thought necessary for my purpose; and by his continual superintendence of the observations, I was able to satisfy myself that they were conducted exactly as I desired, and were worthy of the fullest confidence. I cannot adequately express my sense of the attention which thus put me in possession of the data that I desired, and in the very form in which I desired them, without the smallest trouble to me in the whole transaction.

The mode of making the observations was the following. Under the direction of Captain Shirreff, a vertical scale of feet and inches was marked on the return of the wharf-wall adjoining to the principal landing-stairs of the Victualling Yard. The graduations increased in going downwards, the top of the wharf-wall being the zero. As the bottom of this return of the wall was sometimes dry at low water, a level line was carried to the extremity of the causeway at the bottom of the principal stairs, and another vertical scale (in continuation of the former) was measured there. Thus every observation of the surface of the water was a measure of its depression below the top of the wharf-wall. The times of the observations were in all cases the quarters of hours of mean solar time, as indicated by the striking of the clock of the Victualling Yard. It is proper to mention, that, in consequence of the extensive visibility of the signal-ball of the Royal Observatory (which is dropped at 1^h P.M. precisely), the public clocks in the neighbourhood of Greenwich are for the most part extremely well regulated; and I have therefore little doubt that the times of observation are pretty accurately those which they profess to be.

The object of the first series of observations was simply to ascertain the times of high and low water, for the purpose of ascertaining the durations of the rise and fall of the tide. With this view, the depression of the water below the wharf was observed at the four quarters-of-hour nearest to the time of high water, and at the seven quarters-of-hour nearest to the time of low water (a greater number of observations being made near low water, because less is known, from other observations, of the time of low water). And I proposed, by combining the observations of each group, to find the times of high water and of low water much more accurately than by the rude observation of the highest or the lowest water. But in discussing the low-water observations, I found an unexpected difficulty. The rise of the water at a given interval after low water (in half an hour, for instance) is *considerably* more rapid than its descent at the same interval before low water. There is in fact the rudiment of a veritable *bore*. It is impossible here to use any observations for determining the time of low water except those which are very near to the low water.

My curiosity was now excited to learn with a little more precision the laws of the rise and fall of the water generally. For this purpose, Captain SHIRREFF undertook, at my request, to arrange for the observation of the depression of the water at every quarter-of-an-hour, night and day, during half a lunation. This was done, so far as I can judge, with the most perfect regularity. The observations commenced at February 16, $12^h\ 15^m$, astronomical time, and finished at March 4, $12^h\ 0^m$. The whole number of observations is 1536.

In the computation of these observations it is to be considered that the times of high and low water are subject to perpetual irregularities, as well from the change of conformation of the sun and moon, as from the effects of the wind: and also that the heights are more conspicuously variable from the same cause. But there is this difference between them, as regards the mode of treating the observations on the present occasion. The times of high water at London are predicted with considerable accuracy in the Nautical Almanac: and though the time of high water at Deptford, even when undisturbed, is not the same as at London, yet its difference may be supposed to be nearly constant, and therefore the time of high water at London will be a proper zero of phase to which to refer the Deptford tides. But there is no prediction whatever of the depressions of the Deptford high and low waters; and therefore it appears impracticable to take any zero except the observed least and greatest depressions. These considerations suggested the following methods of reducing the observations:—

The time, between one predicted high water for London and the next, was supposed to correspond to 360° of *phase*; and the interval of each time of Deptford observation from the preceding London predicted time of high water was converted into *phase* by that proportion.

The space, between the least depression and greatest depression in one semi-tide (rise or fall), was supposed to correspond to 2, or the double radius; and the depression of the water at each Deptford observation below the least depression of that semi-tide was converted into parts of 1 or radius by that proportion.

I may remark that the results from the various tides would have agreed more exactly if an observed time of Deptford high water had been used (in the conversion of time into phase) instead of predicted time of London high water: but the process would have been more troublesome, and the advantage very small.

As there was evidently a variation in the law depending on the range of the tide, the observations were next divided into two nearly equal groups; one (of fifteen complete tides) comprising those in which the range was small; the other (of sixteen tides) comprising those in which the range was large. The former is called division A, and the latter division B. These two divisions were treated separately; although the methods of treating them are in every respect the same.

The next step was (in each division) to pick out all the phases included between 0° and 10°, and their corresponding converted depressions; and to take the mean of each of these series of numbers: then to pick out all the phases included between 10° and 20°, and their corresponding converted depressions; and to take the mean of each of these series of numbers: and so on. In this manner there were obtained, in each division, two columns of numbers (every one of which was the mean of about twenty-two numbers); one of them being a series of phases very near to 5°, 15°, &c., and the other being the corresponding converted depression. As there was no difficulty in inferring from these numbers, with considerable accuracy, the change of converted depression corresponding to a very small change of phase, it was easy to apply the correction required to adapt the converted depressions to the exact values of phase 5°, 15°, 25°, &c. In this manner the two following Tables were formed.

Division A.

Mean depression of high water 5 ft. 11 in.
Mean range of tide 15 ft. 3 in.

Phase.	Converted depression.	Phase.	Converted depression.
5°	0·068	185°	1·980
15	0·167	195	1·955
25	0·318	205	1·845
35	0·476	215	1·731
45	0·626	225	1·581
55	0·783	235	1·413
65	0·921	245	1·266
75	1·062	255	1·075
85	1·183	265	0·921
95	1·299	275	0·760
105	1·438	285	0·612
115	1·525	295	0·468
125	1·645	305	0·347
135	1·725	315	0·236
145	1·810	325	0·135
155	1·885	335	0·057
165	1·934	345	0·021
175	1·965	355	0·016

Division B.

Mean depression of high water 3 ft. 6 in.
Mean range of tide 19 ft. 2 in.

Phase.	Converted depression.	Phase.	Converted depression.
5°	0·030	185°	1·912
15	0·118	195	1·960
25	0·249	205	1·981
35	0·406	215	1·923
45	0·563	225	1·753
55	0·697	235	1·568
65	0·830	245	1·366
75	0·942	255	1·156
85	1·057	265	0·988
95	1·171	275	0·830
105	1·272	285	0·683
115	1·371	295	0·557
125	1·469	305	0·454
135	1·558	315	0·347
145	1·645	325	0·232
155	1·725	335	0·116
165	1·794	345	0·042
175	1·851	355	0·008

If we express the numbers in these two columns by series of sines and cosines of multiple arcs, we find the following expressions:—

Division A (neap tides).

$$\begin{aligned}\text{Converted depression} = {} & 1{\cdot}035 \\ & + 0{\cdot}198 \text{ sine phase} - 0{\cdot}918 \cos \text{phase} \\ & + 0{\cdot}064 \text{ sine } 2 \text{ phase} - 0{\cdot}016 \cos 2 \text{ phase} \\ & - 0{\cdot}010 \text{ sine } 3 \text{ phase} - 0{\cdot}047 \cos 3 \text{ phase} \\ & + 0{\cdot}008 \text{ sine } 4 \text{ phase} - 0{\cdot}009 \cos 4 \text{ phase}\end{aligned}$$

$$\begin{aligned}= 1{\cdot}035 & - 0{\cdot}939 \cos (\text{phase} + 12^\circ\, 10') + 0{\cdot}066 \text{ sine } (2 \text{ phase} - 14^\circ\, 2') \\ & - 0{\cdot}048 \cos (3 \text{ phase} - 12^\circ\, 1') + 0{\cdot}012 \text{ sine } (4 \text{ phase} - 48^\circ\, 22').\end{aligned}$$

If we make *phase* $+ 12^\circ\, 10' = p + 90^\circ$, and if we remark that, to obtain the actual depression in feet and inches below the top of the wharf-wall, we must multiply the converted depression by half the range, and must add to the product the depression of high water, we find for the actual depression,

$$5 \text{ ft. } 11 \text{ in.} + \frac{15 \text{ ft. } 3 \text{ in.}}{2} \times \left\{ \begin{array}{l} 1{\cdot}035 + 0{\cdot}939 \,.\, \text{sine } (p) - 0{\cdot}066 \,.\, \text{sine } (2p - 38^\circ\, 22') \\ \quad - 0{\cdot}048 \text{ sine } (3p - 48^\circ\, 40') + 0{\cdot}012 \text{ sine } (4p - 97^\circ\, 2') \end{array} \right\},$$

or

$$13 \text{ ft. } 10 \text{ in.} + 7 \text{ ft. } 7{\cdot}5 \text{ in.} \times \left\{ \begin{array}{l} 0{\cdot}939 \text{ sine } (p) - 0{\cdot}066 \text{ sine } (2p - 38^\circ\, 22') \\ \quad - 0{\cdot}048 \text{ sine } (3p - 48^\circ\, 40') + 0{\cdot}012 \text{ sine } (4p - 97^\circ\, 2') \end{array} \right\},$$

where p represents an angle increasing uniformly with the time and going through a change of 360° in one complete tide.

Division B (spring tides).

$$\begin{aligned}\text{Converted depression} = {} & 1{\cdot}017 \\ & + 0{\cdot}057 \text{ sine phase} - 0{\cdot}900 \text{ cosine phase} \\ & + 0{\cdot}104 \text{ sine } 2 \text{ phase} - 0{\cdot}022 \text{ cosine } 2 \text{ phase} \\ & - 0{\cdot}054 \text{ sine } 3 \text{ phase} - 0{\cdot}043 \text{ cosine } 3 \text{ phase} \\ & + 0{\cdot}016 \text{ sine } 4 \text{ phase} - 0{\cdot}029 \text{ cosine } 4 \text{ phase}\end{aligned}$$

$$\begin{aligned}= 1{\cdot}017 & - 0{\cdot}902 \text{ cosine } (\text{phase} + 3^\circ\, 37') + 0{\cdot}106 \text{ sine } (2 \text{ phase} - 11^\circ\, 57') \\ & - 0{\cdot}069 \,.\, \text{cosine } (3 \text{ phase} - 51^\circ\, 28') + 0{\cdot}033 \text{ sine } (4 \text{ phase} - 61^\circ\, 7').\end{aligned}$$

If we make *phase* $+ 3^\circ\, 37' = p + 90^\circ$, we find as before for the actual depression of the water below the top of the wharf-wall,

$$3 \text{ ft. } 6 \text{ in.} + \frac{19 \text{ ft. } 2 \text{ in.}}{2} \times \left\{ \begin{array}{l} 1{\cdot}017 + 0{\cdot}902 \,.\, \text{sine } (p) - 0{\cdot}106 \text{ sine } (2p - 19^\circ\, 11') \\ \quad - 0{\cdot}069 \text{ sine } (3p - 62^\circ\, 19') + 0{\cdot}033 \text{ sine } (4p - 75^\circ\, 35') \end{array} \right\},$$

or

$$13 \text{ ft. } 3 \text{ in.} + 9 \text{ ft. } 7 \text{ in.} \times \left\{ \begin{array}{l} 0{\cdot}902 \,.\, \text{sine } (p) - 0{\cdot}106 \text{ sine } (2p - 19^\circ\, 11') \\ \quad - 0{\cdot}069 \text{ sine } (3p - 62^\circ\, 19') + 0{\cdot}033 \text{ sine } (4p - 75^\circ 35') \end{array} \right\},$$

where (as before) p represents an angle increasing uniformly with the time, and going through a change of 360° in one complete tide.

On these expressions we may make the following remarks:—

1st. The mean height of water (understanding by the mean height that part of the expression for the height which is independent of sines and cosines of periodical terms) is, at Deptford, not the same for spring tides and for neap tides. The mean height in the average of the high tides is 13 ft. 3 in. below the top of the wharf-wall, and in the average of the low tides is 13 ft. 10 in. below the same point; or the mean height in high tides is greater than the mean height in low tides by seven inches. The corresponding difference in the whole range of the tide is about four feet.

2nd. The curves representing the law of rise and fall of the water are different for high tides and for low tides, and both are sensibly different from the line of sines. This is evident from the algebraical expression, which contains other terms than those depending on the sine and cosine of the simple phase: but it will be more evident to the eye on the comparison of the curves as graphically traced. In Plate I. the strong line represents the law of depression of the surface of the water through every instant of a tide, the horizontal abscissa representing the time or rather the phase, and the vertical ordinate measured downwards representing the depression (as taken from the Table, page 4) for division A (neap tides): the faint line is a line of sines, whose highest point coincides with the highest point of the tide-curve (supposed to have the converted depression 0·013 for phase 350°), and whose lowest point is depressed as far as the lowest point of the tide-curve (supposed to have the converted depression 1·982). In Plate II. the curves are similarly traced for division B; the highest point of the tide-curve is supposed to have the converted depression 0·008 for phase 355°, and the lowest point is supposed to have the converted depression 1·982.

3rd. If we investigate the motion of a very long wave (as a tide-wave) in a rectangular canal whose section is everywhere the same, on the supposition that the extent of vertical oscillation bears a sensible proportion to the mean depth of the water: putting k for the mean depth, $v^2 = g\,k$, and X for the horizontal displacement of any particle (x being its original horizontal ordinate), we find the following partial differential equation:—

$$\frac{d^2 X}{d t^2} = \frac{v^2 \cdot \dfrac{d^2 X}{d x^2}}{\left(1 + \dfrac{d X}{d x}\right)^3}.$$

This equation cannot (it appears) be solved in finite terms, but it may be solved approximately by successive substitution. Putting it in the shape

$$\frac{d^2 X}{d t^2} - v^2 \frac{d^2 X}{d x^2} = v^2 \frac{d^2 X}{d x^2} \times \left\{ -3 \frac{d X}{d x} + 6 \left(\frac{d X}{d x}\right)^2 - \&c. \right\}$$

and first neglecting the second side of the equation entirely, and solving without it; then substituting the solution (adopting that form of function which is adapted to the sea-tide at the mouth of the canal) in the first term on the second side, and solving again; then substituting the solution in the two first terms on the second side and solving again, &c., we find as many terms as we please for X. Then the vertical elevation V of the surface, corresponding to the particle whose original horizontal ordinate was x, is found by the equation $\mathrm{V} = \frac{k}{1 + \frac{d\mathrm{X}}{dx}}$. In order to find the elevation not of a given *particle* but at a given *place*, we must approximately express x, the original ordinate of the particle, in terms of x', the ordinate of the place. After all these operations, putting m for the constant which makes $m\,v\,t$ to change through 360° in one complete tide, and putting $k\,b$ for the coefficient of the first variable term, we find for the elevation of the water,

$$\mathrm{V} = k\left\{1 - b\sin(mvt - mx') + \frac{33}{32}b^3 . mx' . \cos(mvt - mx') + \frac{9}{32}b^3 . m^2x'^2 . \sin(mvt - mx') + \frac{3}{4}b^2 . mx' . \sin(2mvt - 2mx') - \frac{21}{32}b^3 . mx' . \cos(3mvt - 3mx') - \frac{27}{32}b^3 . m^2x'^2 . \sin(3mvt - 3mx')\right\},$$

where the approximation is carried to the third power of b. This expression supposes the canal unlimited at the end furthest from the sea. If the canal is stopped by a barrier, the expression changes its form. Putting a for the distance of the barrier from the sea, the elevation at the point x' is represented by

$$\mathrm{V} = k\left\{1 - \frac{c . \cos(ma - mx')}{\cos ma}\sin mvt + \frac{c^2}{8\cos^2 ma}\left(\cos(2ma - 2mx') - \cos 2ma\right) + \frac{3 . c^2 . mx' . \sin(2ma - 2mx')}{8\cos^2 ma}\cos 2mvt\right\},$$

or, putting $k\,b$ for the coefficient of the first variable term, and omitting that term which does not vary with the time,

$$\mathrm{V} = k\left\{1 - b\sin mvt + \frac{3}{4}b^2 . mx' . \tan(ma - mx') . \cos 2mvt\right\},$$

where the approximation is carried to the second power of b. Neither of the supposed circumstances corresponds exactly to the case of a tidal river; but it may with some reason be supposed to be represented by something intermediate to them; the bridges and other impediments in the upper part producing in some degree the same effect as a barrier. The slope of the sides of the channel alters the magnitude of the coefficients, but does not appear to alter the general form of the expressions: the investigation, however, though not difficult, is so troublesome that I have not completed it. Thus from theory we should expect the variable part of the converted depression to be expressed by a formula intermediate to the two following, the multiplier $k\,b$ being omitted:

$$+\sin(mvt - mx') - \frac{3}{4}b . mx' . \sin(2mvt \quad 2mx'),$$

$$+\sin mvt - \frac{3}{4}b . mx' . \tan(ma - mx') . \cos 2mvt.$$

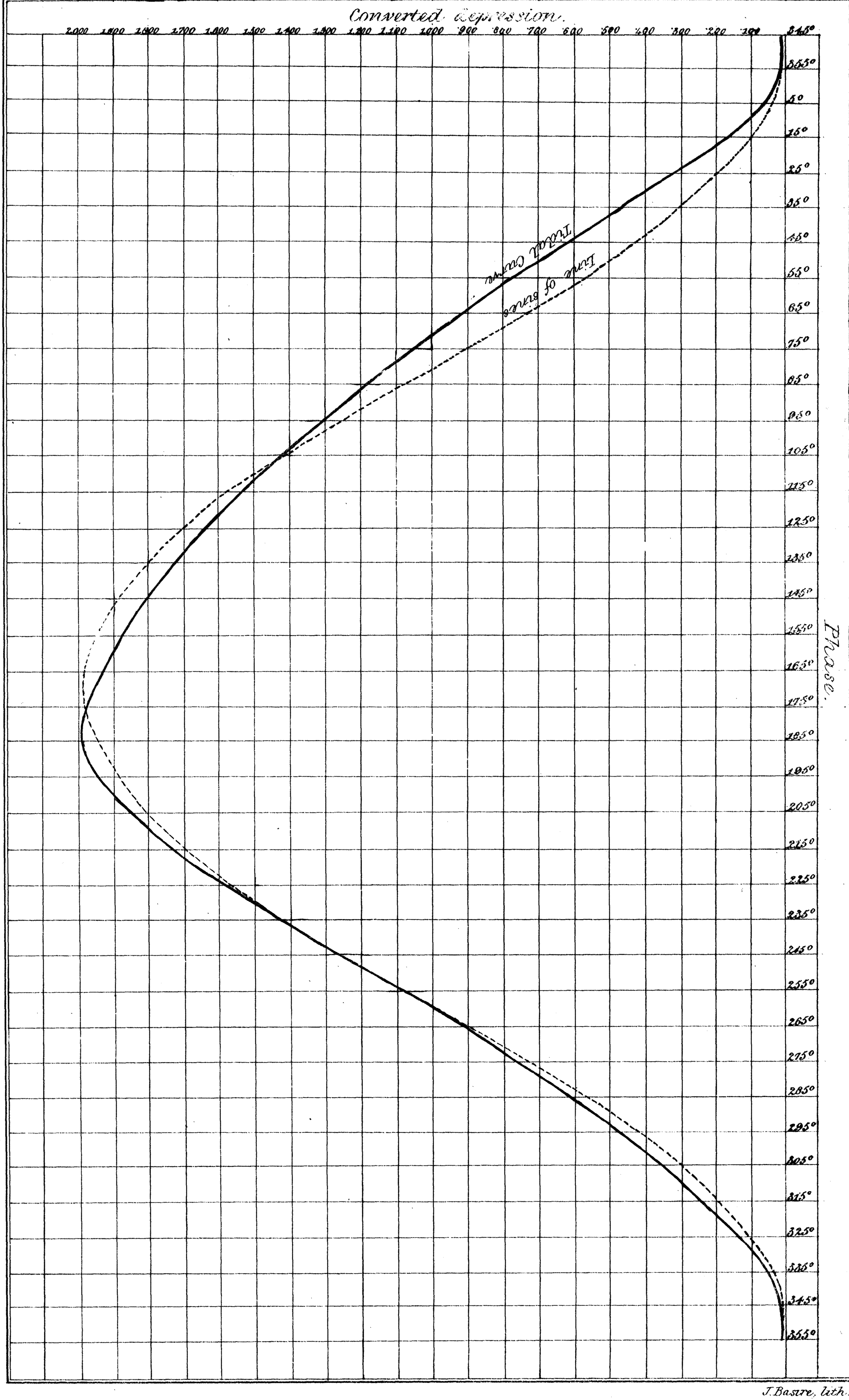

Fig. 1. Curve representing the law of fall and rise of the tide at Deptford, when the whole range of tide is 15 feet 3 Inches.

J. Basire, lith.

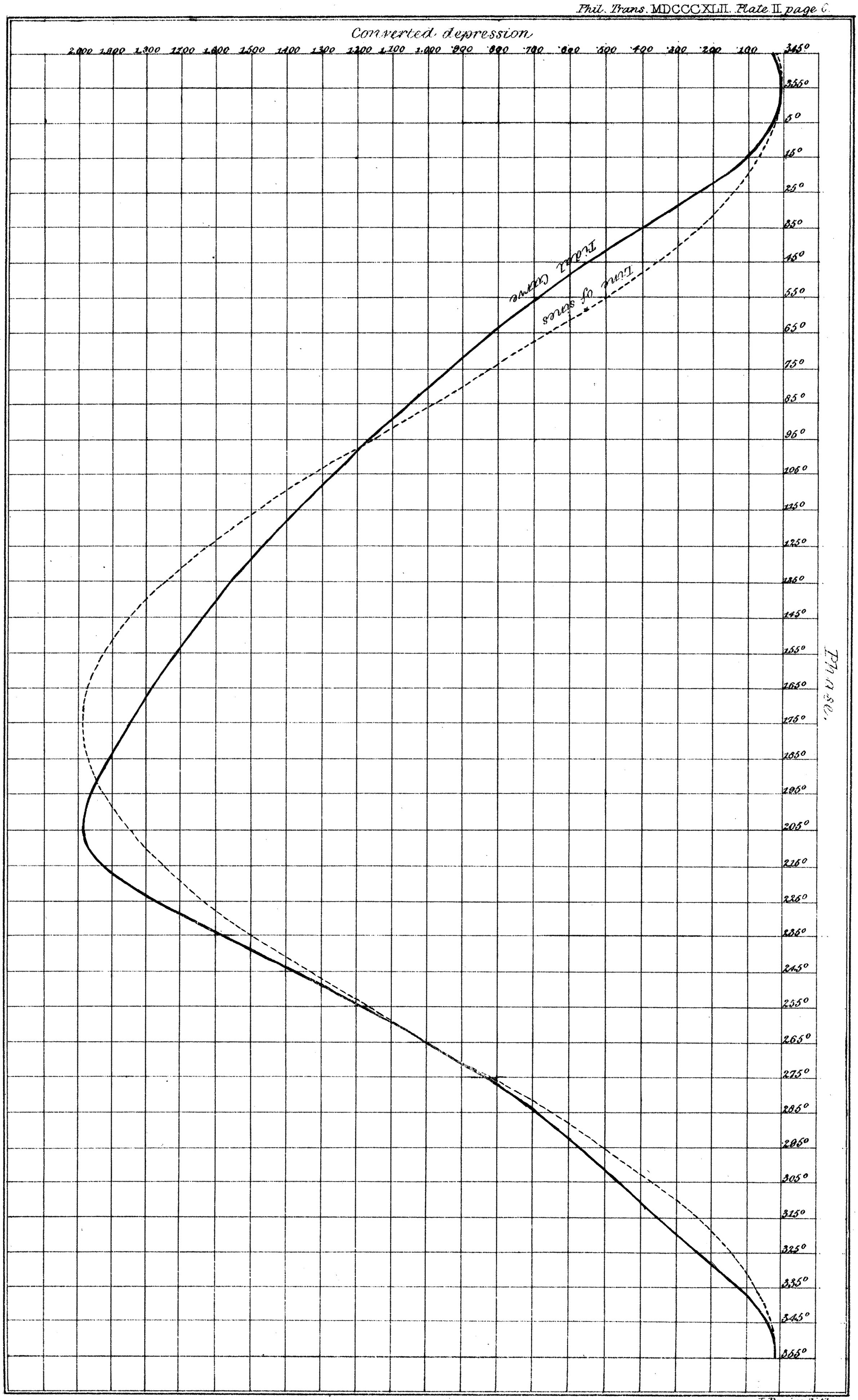

Fig. 2. Curve representing the law of fall and rise of the tide at Deptford, when the whole range of tide is 19 feet 2 inches.

J. Basire, lith.

To see clearly the import of this, suppose that our intermediate formula is to be half the sum of these two expressions. Its form then becomes

$$+\cos\frac{mx'}{2}\cdot\sin\left(mvt-\frac{mx'}{2}\right)$$

$$-\frac{3}{8}b.mx'\left\{\cos 2mx'.\sin 2mvt-\left(\sin 2mx'-\tan(ma-mx')\right)\cos 2mvt\right\}.$$

Let

$$\frac{\sin 2mx'-\tan(ma-mx')}{\cos 2mx'}=\tan(2mx'-\mathrm{D}),$$

then the expression may be put in the form

$$+\mathrm{B}\sin\left(mvt-\frac{mx'}{2}\right)-\mathrm{C}\sin(2mvt-2mx'+\mathrm{D}),$$

or, if $mvt-\frac{mx'}{2}=p$, the theoretical converted depression is

$$+\mathrm{B}\sin p-\mathrm{C}\sin(2p-mx'+\mathrm{D}).$$

When $ma-mx'$ is not large (as it certainly cannot be at Deptford), mx' is greater than D, and the theoretical converted depression, putting A for $mx'-\mathrm{D}$, is

$$+\mathrm{B}.\sin p-\mathrm{C}.\sin(2p-\mathrm{A}).$$

This is precisely the same form as that given by the discussion of the observations.

4th. According to the theory, the coefficient of the second variable term in the expression for the converted depression ought to be proportional to the range of the tide (for it contains b as a factor). In the discussion of the observations it appears that this coefficient increases more rapidly than b. This seems to render it probable that terms of sensible magnitude would be introduced by pushing the approximation to the fourth and higher orders.

5th. Suppose then that the variable part of the converted depression is represented by $\sin\theta-c\sin(2\theta-\mathrm{A})$, where θ, as appears from the theory, is $mvt-mx'$, or is the value of the phase depending simply on the depth of the canal and the distance of the point of observation from the sea, or is that value of phase which would correspond to a shallow wave passing along the canal. The values of θ for high and low water will be those which satisfy the equation $\cos\theta-2c.\cos(2\theta-\mathrm{A})=0$. Solving this equation for high water, by successive substitution, we have as a first approximation $\cos\theta=0$, or $\theta=270°$; $\cos(2\theta-\mathrm{A})=\cos(540°-\mathrm{A})=-\cos\mathrm{A}$; using this substitution in the second term, the equation becomes $\cos\theta+2c\cos\mathrm{A}=0$; or if $\theta=270°+x$, $\sin x+2c\cos\mathrm{A}=0$, or $x=-2c.\cos\mathrm{A}$ nearly, and therefore the value of $mvt-mx'$ for high water is $270°-2c\cos\mathrm{A}$ nearly. In the same way it is found that the value of $mvt-mx'$ for low water is $90°+2c\cos\mathrm{A}$ nearly. So far, therefore, as depends on this term, the high water is accelerated and the low water is retarded by nearly equal terms: and this acceleration and retardation are proportional to c, or to b, or to the whole range of the tide: and are therefore greater for spring tides than for neap tides.

6th. This remark enables us to explain a circumstance which appeared somewhat perplexing. It has been found by Sir J. W. LUBBOCK and Mr. WHEWELL that the age of the tide is different, as inferred from the *height* of the high water, or from the *time* of high water: the age of the tide always appearing greater as inferred from the *heights**. Now to explain this, we have to consider that, at syzygies of the sun and moon, the time of high water in the sea is on every successive day earlier with respect to the moon's transit; but the syzygial or spring tide in the river, and the tides near it, are (as we have just found) accelerated with respect to the sea-tide more than mean tides are: and therefore the river tide which happens at the mean interval from the moon's transit is not the syzygial tide, but a tide preceding it. But there is no corresponding effect produced on the height of the tide. Thus the age of the tide inferred from the height is the true age (at least as far as it can be ascertained from the phenomena of the ocean-tides); the age as inferred from the time of high water is certainly too small, and the quantity by which it is too small depends on the length and depth of the river, or of the shallows along which the tide has to pass.

I have only to add the following deductions from the observations.

In division A (low tides) the high water occurred at the phase 350° nearly, that is, about 20^{m} before the predicted time of high water at London Bridge: the low water occurred at the phase 185° nearly: the interval between high water and low water was about 195° of phase, and that between low water and high water about 165° of phase; or the descent occupied a longer time than the ascent by 30° of phase, or a little more than an hour of time.

In division B (high tides) the high water occurred at the phase 355° nearly, or about 10^{m} before the predicted time of high water at London Bridge: the low water occurred at the phase 205° nearly; the descent therefore occupied 210° of phase, and the ascent 150°; or the time of descent exceeded that of ascent by 60° of phase, or $2^{h}\ 4^{m}$ of time.

The times of the turn of the tide-current, as shown by the swinging of the ships at anchor in the river, were regularly observed. The means of the corresponding phases in division A are 10°·4 and 204°·4, or nearly 20° of phase or 40^{m} of time after high and low water respectively: those in division B are 14°·0 and 223°·5, or nearly 18°·5 of phase or 37^{m} of time after high and low water respectively.

* LUBBOCK, Philosophical Transactions, 1837. WHEWELL, Philosophical Transactions, 1838.

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